



# Kramers-Wannier dualities for WZW theories and minimal models

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## Abstract

We study Kramers-Wannier dualities for Wess-Zumino-Witten theories and (super-)minimal models in the Cardy case, i.e. the case with bulk partition function given by charge conjugation. Using the TFT approach to full rational conformal field theories, we classify those dualities that preserve all chiral symmetries. Dualities turn out to exist for small levels only.

# 1 Introduction

Two-dimensional conformal field theories have been an essential tool to study universal properties of critical phenomena. They capture surprisingly many aspects of statistical models with critical points, even away from criticality. A fascinating aspect of some of these models are Kramers-Wannier like dualities, relating e.g. the high-temperature and low-temperature regime; the critical point is typically self-dual. While this has been known for more than sixty years [15], the obvious question whether such dualities can be deduced from properties at the critical point has only been addressed recently.

The – affirmative – answer uses <sup>1</sup> an algebraic approach to full (rational) conformal field theories [8, 9] that describes correlation functions of these theories in terms of two pieces of data:

- the chiral data of the conformal field theory, which are encoded in a modular tensor category  $\mathcal{C}$
- a (symmetric special) Frobenius algebra  $A$  in the tensor category  $\mathcal{C}$ .

For the purposes of the present paper, a modular tensor category  $\mathcal{C}$  (see [20] and e.g. [1]) is defined to be a semi-simple  $\mathbb{C}$ -linear abelian ribbon category with simple tensor unit  $\mathbb{1}$ , having a finite number of isomorphism classes of simple objects; the braiding on the tensor category is required to obey a certain nondegeneracy condition. (This definition is slightly more restrictive than the original one in [20].)

In the TFT approach to rational conformal field theory, types of topological defect lines correspond to isomorphism classes of  $A$ -bimodules. Given two bimodules  $B_1$  and  $B_2$ , their tensor product  $B_1 \otimes_A B_2$  is again a bimodule; this tensor product encodes the fusion of topological defects. In the same way the modular tensor category  $\mathcal{C}$  of chiral data gives rise to a fusion ring  $K_0(\mathcal{C})$  of chiral data, the tensor category  $\mathcal{C}_{AA}$  of  $A$ -bimodules gives rise to a fusion ring  $K_0(\mathcal{C}_{AA})$  of topological defects. Both fusion rings are semi-simple. The fusion ring of defects is, however, not necessarily commutative, since the category of bimodules is typically not braided. (For a more detailed discussion of the fusion ring  $K_0(\mathcal{C}_{AA})$  see e.g. [10].)

The present paper builds on the insight of [3, 4] that the fusion ring  $K_0(\mathcal{C}_{AA})$  of topological defects determines both symmetries and Kramers-Wannier dualities of the full conformal field theory described by the pair  $(\mathcal{C}, A)$ . The fact that Kramers-Wannier dualities relate [13] bulk fields to disorder fields located at the end points of topological defect lines might have suggested a relation between Kramers-Wannier dualities and topological defects. However, the topological defect lines relevant for the dualities are of

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<sup>1</sup>See also [16] for a different approach.

a different type than the ones created by the dual disorder fields.

We explain the pertinent results of [3, 4] in more detail: symmetries of the full conformal field theory  $(\mathcal{C}, A)$  correspond to isomorphism classes of invertible objects in  $\mathcal{C}_{AA}$ . These are objects  $B$  satisfying

$$B \otimes_A B^\vee \cong A, \quad B^\vee \otimes_A B \cong A, \quad (1)$$

where  $B^\vee$  is the bimodule dual to  $B$ . The isomorphism classes of the invertible bimodules form a group, the Picard group  $\text{Pic}(\mathcal{C}_{AA})$ . This group is not necessarily commutative: for the three state Potts model, for example, it turns out to be the symmetric group  $S_3$  on three letters, see [3].

To describe dualities, we need the following

**Definition 1.1.** *Given a modular tensor category  $\mathcal{C}$  and a simple symmetric special Frobenius algebra  $A$  in  $\mathcal{C}$ , a simple  $A$ -bimodule  $B$  is called a duality bimodule [3] iff all simple subobjects of the tensor product  $B^\vee \otimes_A B$  are invertible bimodules.*

It is easy to see that for any duality bimodule  $B$  the isomorphism classes of simple bimodules  $B_\lambda$  such that  $\dim_{\mathbb{C}} \text{Hom}(B_\lambda, B^\vee \otimes_A B) > 0$  form a subgroup  $H$  of the Picard group  $\text{Pic}(\mathcal{C}_{AA})$  and that in this case  $\dim_{\mathbb{C}} \text{Hom}(B_\lambda, B^\vee \otimes_A B) = 1$ . We call  $H$  the *stabilizer* of the duality bimodule  $B$ .

In the present paper, we restrict ourselves to the case where the simple symmetric special Frobenius algebra in  $\mathcal{C}$  is the tensor unit  $\mathbb{1}$ ; this situation is usually referred to as the Cardy case. A complete set of correlation functions for the full conformal field theory in the Cardy case has been constructed in [2]. In the Cardy case, the bulk partition function is given by charge conjugation. Many more simplifications occur: The category of  $A$ -bimodules is equivalent to the original category,  $\mathcal{C}_{AA} \cong \mathcal{C}$ ; as a consequence, isomorphism classes of invertible bimodules are simple currents [18]:

**Definition 1.2.** (i) *A simple current in a modular tensor category  $\mathcal{C}$  is an isomorphism class  $[J]$  of simple objects  $J$  satisfying*

$$J \otimes J^\vee \cong \mathbb{1}. \quad (2)$$

(ii) *A fixed point  $[U_\phi]$  of a simple current  $[J]$  is an isomorphism class of simple objects of  $\mathcal{C}$  satisfying*

$$J \otimes U_\phi \cong U_\phi. \quad (3)$$

(iii) *A duality class is an isomorphism class  $[U_\phi]$  of simple objects such that the tensor product  $U_\phi^\vee \otimes U_\phi$  is isomorphic to a direct sum of invertible objects containing at least two non-isomorphic invertible objects.*

As a special case of the results of [3, 4], we see that the full conformal field theory in the Cardy case has Kramers-Wannier dualities if and only if the underlying modular tensor category has duality classes.

In the present paper, we study two classes of unitary rational conformal field theories: (super-)Virasoro minimal models and Wess-Zumino-Witten (WZW) theories. There is a WZW theory for every reductive finite-dimensional complex Lie algebra; in the present paper, we limit ourselves to simple Lie algebras. Thus, both classes of conformal field theories come in families that are parametrized by a positive integer, called the level.

Simple currents in Virasoro minimal models (and thus their symmetries) are well known; for WZW theories, simple currents have been classified in [6]: with the exception of the WZW theory based on  $E_8$  at level 2, they are in bijection with the center of the corresponding simple, connected, simply-connected compact Lie group. It might be surprising at first sight that only the center - rather than at least the full Lie group - shows up as the symmetry group. One should keep in mind, however, that the symmetries we discuss are required to preserve all chiral symmetries, i.e. the complete current algebra. Relaxing this requirement - which amounts to working with the representation category of a subalgebra of the current algebra and a nontrivial symmetric special Frobenius algebra in this category - leads to larger symmetry groups.

In this paper, we present a classification of Kramers-Wannier dualities in these models. To state our results, we need to introduce some notation: In WZW theories the chiral algebra is generated by an untwisted affine Lie algebra  $X_r^{(1)}$  of rank  $r$ ; moreover, one has to fix a positive integral value  $k$  of the level. We denote the corresponding modular tensor category by  $\mathcal{C}[(X_r)_k]$ .

In this paper we establish the following results:

**Theorem 1.3.** *The only Wess-Zumino-Witten theories  $(\mathcal{C}[(X_r)_k], \mathbb{1})$  with duality classes are  $(E_7)_2, (A_1)_2, (B_r)_1$  and  $(D_{2r})_2$  with  $r \geq 2$ .*

Since the finite-dimensional complex simple Lie algebras  $B_2$  and  $C_2$  are isomorphic, we do not list  $(C_2)_1$  separately. It should be noted that dualities only appear at low level. We are, unfortunately, not aware of any a priori argument for this finding. For all cases except  $(D_{2r})_2$ , the stabilizer is cyclic of order two. For most of the cases, the existence of the dualities does not come as a surprise: the modular tensor categories for  $(A_1)_2$  and  $(B_r)_1$  have the same fusion rules as the Ising model. For  $(E_7)_2$  the fusion rules are isomorphic to those of the tricritical Ising model and hence the category contains a tensor subcategory with Ising fusion rules.

Concerning Virasoro minimal models and their superconformal counterparts, we find the following situation.

**Theorem 1.4.** (i) *Duality classes only exist for the unitary Virasoro minimal models at level  $k = 1$  and  $k = 2$ , i.e. for the Ising model and the tricritical Ising model.*

(ii) *Duality classes for an  $N = 1$  super-Virasoro unitary minimal model only exist for odd levels. If they exist, they are unique.*

(iii)  *$N = 2$  superconformal minimal models have duality classes only for level  $k = 2$ .*

Our findings agree, for the nonsupersymmetric theories, with the ones of [16]; our methods, however, are different. Again we find a close relation to Ising fusion rules: all models with a Kramers-Wannier duality have a realization by a coset construction involving  $(A_1)_2$ .

The plan of the paper is as follows: Section 2 contains model independent results. We analyze properties of duality classes; in Theorem 2.2.1 we give a necessary condition for the existence of duality classes that is used in the analysis of the  $A$ -series of Wess-Zumino-Witten models and the minimal models.

For the  $B$ -,  $C$ -,  $D$ -series and the exceptional algebras  $E_6$  and  $E_7$ , there are no simple objects meeting the conditions of Theorem 2.2.1. A different strategy involving lower bounds of quantum dimensions is developed in Section 2; it is based on the results of [11] on second-lowest quantum dimensions in Wess-Zumino-Witten fusion rules. Section 3 contains the analysis of dualities for Wess-Zumino-Witten theories; Section 4 is devoted to the study of (super-)conformal minimal models.

## 2 Model-independent considerations

### 2.1 Preliminary remarks and notation

We start by introducing some notation: we choose a set  $(U_\lambda)_{\lambda \in I}$  of representatives for the isomorphism classes of simple objects of the modular tensor category  $\mathcal{C}$ . In particular, given  $\mu \in I$ , we find a unique  $\bar{\mu}$  such that  $U_\mu^\vee \cong U_{\bar{\mu}}$ . The tensor unit  $\mathbb{1}$  of a modular tensor category is simple; we choose the representatives such that  $U_0 = \mathbb{1}$  and  $0 \in I$ .

The classes  $[U_\lambda]$  form the distinguished basis of the fusion ring  $K_0(\mathcal{C})$ . The fusion coefficients

$$\mathcal{N}_{\lambda\mu}^\rho := \dim_{\mathbb{C}} \text{Hom}(U_\lambda \otimes U_\mu, U_\rho)$$

are the structure constants of the multiplication on  $K_0(\mathcal{C})$ . The following identities are easy consequences of the properties of a duality of a tensor

category:

$$\mathcal{N}_{\lambda\mu}^{\rho} = \mathcal{N}_{\mu\bar{\rho}}^{\bar{\lambda}} = \mathcal{N}_{\bar{\rho}\lambda}^{\bar{\mu}}. \quad (4)$$

The braiding isomorphism  $c_{U,V} : U \otimes V \rightarrow V \otimes U$  of a modular tensor category furnishes a symmetric matrix  $s_{ij} = \text{tr} c_{U_j, U_i} \circ c_{U_i, U_j}$ . For the models considered in this paper, there exists a positive real factor such that the matrix  $s$  rescaled by this factor is unitary. We call this unitary matrix  $S$ . The fusion coefficients can be expressed in terms of the modular matrix  $S$  by the Verlinde formula

$$\mathcal{N}_{\rho\sigma}^{\tau} = \sum_{\kappa \in I} \frac{S_{\rho\kappa} S_{\sigma\kappa} \bar{S}_{\tau\kappa}}{S_{0\kappa}}. \quad (5)$$

Another property of the modular matrices  $S$  in the models of our interest are the inequalities  $S_{0\kappa} \geq S_{00} > 0$  for any simple object  $U_{\kappa}$ . Entries for dual objects are related by complex conjugation,  $S_{\lambda\bar{\mu}} = \overline{S_{\lambda\mu}}$ . We finally note the following easy consequence of the Verlinde formula (5) and the unitarity of  $S$

$$\frac{S_{\rho\kappa}}{S_{0\kappa}} \frac{S_{\sigma\kappa}}{S_{0\kappa}} = \sum_{\tau \in I} \mathcal{N}_{\rho\sigma}^{\tau} \frac{S_{\tau\kappa}}{S_{0\kappa}}. \quad (6)$$

We need to introduce two more notions for modular tensor categories. Recall the definition of a simple current from the introduction. The set of simple currents carries the structure of a finite abelian group, called the Picard group  $\text{Pic}(\mathcal{C})$ . It turns out that every isomorphism class  $[U_{\kappa}]$  of simple objects of  $\mathcal{C}$  gives rise to a character on the group  $\text{Pic}(\mathcal{C})$ :

$$\chi_{\lambda}([U_{\kappa}]) := \frac{S_{\lambda\kappa}}{S_{0\lambda}}, \quad (7)$$

which we call the monodromy character of the object  $U_{\kappa}$ .

A modular tensor category being in particular a ribbon tensor category, there is the notion of a (quantum) dimension for any object  $U$  of  $\mathcal{C}$ . It depends only on the isomorphism class of an object. For simple objects, we introduce the abbreviation

$$\mathcal{D}_{\lambda} := \dim(U_{\lambda}).$$

The quantum dimension is related to the modular  $S$ -matrix via

$$\mathcal{D}_{\lambda} = \frac{S_{0\lambda}}{S_{00}}. \quad (8)$$

The quantum dimension is a ring homomorphism from  $K_0(\mathcal{C})$  to the ring of algebraic integers over the rational numbers. Dual objects have identical dimension

$$\mathcal{D}_{\bar{\lambda}} = \mathcal{D}_{\lambda}.$$

Since the quantum dimensions of the modular tensor categories we consider are the Frobenius-Perron eigenvalues of their fusion matrices, they are real and obey  $\mathcal{D}_{\lambda} \geq 1$ . Equality is achieved precisely for simple currents.

It follows immediately from the properties of the quantum dimension that the quantum dimension of a duality object of  $\mathcal{C}$  is the square root of an integer  $\mathcal{D}_{\phi} = \sqrt{|H|}$ , with  $|H|$  the order of the stabilizer  $H \leq \text{Pic}(\mathcal{C})$ .

To discuss WZW theories, we finally need some Lie-theoretic notation. Let  $X_r^{(1)}$  be an untwisted affine Lie algebra. Denote by  $\Lambda^i$  the fundamental weights and by  $a_i^{\vee}$  the dual Coxeter labels. Their sum equals the dual Coxeter number,  $g^{\vee} = \sum_{i=0}^r a_i^{\vee}$ . A labelling of the nodes of the Dynkin diagram provides a labelling of simple roots and fundamental weights; we use the conventions of [14, 5].

Fix a nonnegative integer  $k$ , the level. At level  $k$ , there are finitely many integrable highest weights  $\lambda$

$$\lambda \in P_+^k = \{(\lambda_0, \lambda_1, \dots, \lambda_r) := \sum_{i=0}^r \lambda_i \Lambda^i \mid \lambda_i \in \mathbb{Z}_{\geq 0}, \sum_{i=0}^r a_i^{\vee} \lambda_i = k\}.$$

Isomorphism classes of simple objects of the modular tensor category  $\mathcal{C}[(X_r)_k]$  are in bijection to elements of  $P_+^k$ ; in particular, the irreducible highest weight representation with highest weight  $k\Lambda_0$  is the tensor unit  $U_0$ .

For fixed level  $k$ , the zeroth component of a highest weight is redundant. We therefore work with the finite-dimensional simple Lie algebra  $\bar{X}_r$ , called the horizontal subalgebra, and the horizontal part  $\bar{\lambda} = \sum_{i=1}^r \lambda_i \Lambda^i$  of the weight  $\lambda$ . The quantum dimension of the simple object  $U_{\lambda}$  can be expressed by a deformed version of Weyl's dimension formula in terms of a product over a set of positive roots of the horizontal subalgebra  $\bar{X}_r$ :

$$\mathcal{D}_{\lambda}^{(k)} = \prod_{\bar{\alpha} > 0} \frac{[(\bar{\lambda} + \bar{\rho}, \bar{\alpha})]_k}{[(\bar{\rho}, \bar{\alpha})]_k}. \quad (9)$$

Here  $\bar{\rho} = \sum_{i=1}^r \Lambda^i$  is the Weyl vector of  $\bar{X}_r$  and for given level  $k \in \mathbb{N}$ , the bracket  $[x]_k$  of a rational number  $x$  is the real number

$$[x]_k := \sin \left( \frac{\pi x}{k + g^{\vee}} \right).$$

The identity

$$\lfloor x \rfloor_k = \lfloor k + g^\vee - x \rfloor_k . \quad (10)$$

is immediate.

## 2.2 A criterion for the existence of dualities

We start with the following useful criterion:

**Theorem 2.2.1.** *Let  $H \leq \text{Pic}(\mathcal{C})$  be a subgroup of simple currents. Suppose that there is a simple object  $U_\mu$  with the following two properties:*

1. *The restriction of the monodromy character  $\chi_\mu$  of  $U_\mu$  to  $H$  is nontrivial.*
2. *The tensor product of  $U_\mu$  with its dual object contains, apart from the tensor unit, just one more simple object:*

$$U_\mu \otimes U_\mu^\vee \cong \mathbb{1} \oplus U_\nu .$$

*Then a duality class with stabilizer  $H$  can only exist if the simple object  $U_\nu$  appearing in the tensor product  $U_\mu \otimes U_\mu^\vee$  is a non-trivial simple current.*

Proof. Suppose  $U_\phi$  is a duality class; then formula (6) gives for any  $\kappa \in I$ :

$$\left| \frac{S_{\phi\kappa}}{S_{0\kappa}} \right|^2 = \frac{S_{\phi\kappa}}{S_{0\kappa}} \frac{S_{\bar{\phi}\kappa}}{S_{0\kappa}} = \sum_{J \in H} \frac{S_{J\kappa}}{S_{0\kappa}} ,$$

where  $H$  is the stabilizer of  $U_\phi$ .

The definition of the monodromy character (7) together with standard properties of characters of finite abelian groups implies

$$\left| \frac{S_{\phi\kappa}}{S_{0\kappa}} \right|^2 = \sum_{J \in H} \chi_\kappa(J) . \quad (11)$$

This expression is nonvanishing iff the restriction of the monodromy character of  $U_\kappa$  to  $H$  is trivial; in this case it equals the order  $|H|$  of the group  $H$ . Thus  $S_{\phi\kappa} = 0$ , whenever  $U_\phi$  is a duality class and  $U_\kappa$  a simple object whose monodromy character restricted to  $H$  is nontrivial.

The second property of the simple object  $U_\mu$  together with (6) immediately gives

$$\left| \frac{S_{\mu\phi}}{S_{0\phi}} \right|^2 = 1 + \frac{S_{\nu\phi}}{S_{0\phi}} .$$



By the first assumption,  $U_\mu$  has nontrivial monodromy character; therefore  $S_{\mu\phi} = 0$ . As a consequence,

$$1 + \frac{S_{\nu\phi}}{S_{0\phi}} = 0. \quad (12)$$

Next, we notice that due to its appearance in the tensor product  $U_\mu \otimes U_\mu^\vee$ , the object  $U_\nu$  has necessarily trivial monodromy character. Therefore, relation (11) yields

$$\left| \frac{S_{\phi\nu}}{S_{0\nu}} \right|^2 = |H| \text{ for } U_\nu \text{ and } \left| \frac{S_{\phi 0}}{S_{00}} \right|^2 = |H| \text{ for the tensor unit } U_0.$$

Taking the quotient of the last two relations gives

$$\left| \frac{S_{\phi\nu}}{S_{\phi 0}} \right|^2 = \left| \frac{S_{0\nu}}{S_{00}} \right|^2 = \mathcal{D}_\nu^2.$$

Equation (12) now implies that the left hand side of this equation equals one; hence  $\nu$  has to be a simple current.  $\square$

**Remark 2.2.2.** *Since  $[U_\nu]$  is a simple current, the simple object  $U_\mu$  is itself a duality class for the cyclic stabilizer generated by  $[U_\nu]$ . The simple current  $[U_\nu]$  has order two.*

The criterion of Theorem 2.2.1 will be applied to the  $A$ -series and the minimal models in Section 3.

## 2.3 Monotonicity of quantum dimensions

Let  $[J]$  be a simple current and  $\phi \in I$ . Due to the relation

$$\mathcal{N}_{\phi\bar{\phi}}^J = \mathcal{N}_{J\phi}^\phi$$

the simple current  $J$  appears in the decomposition of the tensor product  $U_\phi \otimes U_\phi^\vee$  if and only if  $[U_\phi]$  is a fixed point of  $[J]$ . Duality classes are thus, in particular, fixed points under a subgroup  $H$  of the Picard group. A fixed point under a subgroup  $H$  of the Picard group is a duality class with stabilizer  $H$  if and only if its quantum dimension equals  $\sqrt{|H|}$ .

To get constraints on the existence of duality classes, we study the growth of the quantum dimension of fixed points of subgroups of the Picard group as a function of the level. The following lemma (cf. also [7]) plays a key role for finding duality classes for WZW theories based on untwisted affine Lie algebras in the  $B$ -,  $C$ -,  $D$ - series and exceptional Lie algebras.

**Lemma 2.3.1.** *Let  $X_r^{(1)}$  be an untwisted affine Lie algebra. Let  $\lambda(k)$  be a family of integral highest weights of  $X_r^{(1)}$  at level  $k$  of the form*

$$\lambda(k) = \sum_{i=1}^r \lambda_i \Lambda^i + (k - \sum_{i=1}^r a_i^\vee \lambda_i) \Lambda^0 ,$$

*i.e. where only the zeroth component of  $\lambda(k)$  depends on the level. Assume that not all  $\lambda_i$  vanish. Denote by  $\mathcal{D}_\lambda(k)$  the quantum dimension of  $\lambda(k)$  at level  $k \in \mathbb{Z}_{\geq 0}$ . Then the expression of equation (9) for  $\mathcal{D}_\lambda(k)$  defines an analytic function on  $\mathbb{R}_{\geq 0}$  which is strictly monotonically increasing.*

Proof. Differentiating equation (9) with respect to  $k$  yields

$$\frac{\partial}{\partial k} \mathcal{D}_\lambda(k) = \mathcal{D}_\lambda(k) \sum_{\bar{\alpha} > 0} (f_k((\rho, \bar{\alpha})) - f_k((\bar{\lambda} + \rho, \bar{\alpha}))) , \quad (13)$$

where the function

$$f_k(x) := \frac{\pi x}{(k + g^\vee)^2} \cot\left(\frac{\pi x}{k + g^\vee}\right)$$

is strictly monotonically decreasing as a function of  $x$  for all fixed levels  $k$ . Thus all summands are nonnegative; moreover,  $(\bar{\lambda}, \bar{\alpha}) > 0$  for at least one positive root  $\bar{\alpha}$ . Hence the expression (13) is positive.  $\square$

### 3 Dualities for WZW theories

The modular tensor categories based on the Lie algebras  $E_8$  for level greater or equal to three,  $F_4$  and  $G_2$  do not have non-trivial simple currents [6]. As a consequence, no duality classes exist. At level two,  $E_8$  has Ising fusion rules and thus has a duality class with cyclic stabilizer of order two.

In the sequel we will use the slightly redundant notation  $\mathcal{D}_\lambda^{(k,r)}$  for the quantum dimension of the weight  $\lambda$  of the algebra  $(X_r)_k$  at level  $k$ . Simple objects will be referred to by the horizontal part of their weight; for reasons of simplicity, we will drop overlines over horizontal weights.

#### 3.1 The affine Lie algebra $E_7$

The tensor categories based on  $E_7$  have cyclic Picard group of order two, generated by the irreducible highest weight representation with highest weight  $k\Lambda^6$ . We are thus led to classify fixed points of quantum dimension  $\sqrt{2}$ . The

action of a simple current on highest weights corresponds to a symmetry of the Dynkin diagram; the nodes invariant under the symmetry corresponding to the nontrivial simple current all have even Coxeter labels. Since fixed points correspond to highest weights invariant under this symmetry, they – and hence duality classes – only occur at even level.

According to [11], the second smallest quantum dimension for given level  $k \geq 5$  occurs for horizontal weight  $\Lambda^6$ ,  $\mathcal{D}_\lambda^{(k)} \geq \mathcal{D}_{\Lambda^6}^{(k)}$ . The monotonicity lemma 2.3.1 yields

$$\mathcal{D}_{\Lambda^6}^{(k)} \geq \mathcal{D}_{\Lambda^6}^{(5)} = \frac{[10]_5}{[1]_5} > \sqrt{2}.$$

Thus, there are no duality classes for level  $k \geq 5$ .

The simple objects of second lowest quantum dimension at level  $k = 2, 4$  have been listed in Table 3 of [11]. At level  $k = 2$ , one finds  $\Lambda^7$  which is a fixed point of quantum dimension

$$\mathcal{D}_{\Lambda^7}^{(2)} = \frac{[2]_2[6]_2[8]_2}{[3]_2[4]_2[5]_2[7]_2} = \sqrt{2}$$

and thus a duality class. Indeed, the fusion rules of  $E_7$  at level 2 are isomorphic to the fusion rules of the tricritical Ising model which is known to exhibit a Kramers-Wannier duality.

At level 4, according to the same table, the second smallest quantum dimension is assumed for  $2\Lambda^7$ . The quantum dimension is larger than  $\sqrt{2}$  (this can be shown e.g. by using the computer program KAC [19]). We conclude that WZW theories based on the untwisted affine Lie algebra  $E_7^{(1)}$  exhibit a duality class only at level two.

### 3.2 The affine Lie algebra $E_6$

The Picard groups of the tensor categories based on  $E_6$  are cyclic of order three. Duality objects are therefore precisely the fixed points of quantum dimension  $\sqrt{3}$ . It follows from the values of the Coxeter labels that they can only occur at levels divisible by three.

According to [11], for  $k \geq 3$ , the second smallest quantum dimension occurs for the weight  $\Lambda^1$ ,  $\mathcal{D}_\lambda^{(k)} \geq \mathcal{D}_{\Lambda^1}^{(k)}$ . By monotonicity in  $k$ , we derive a lower bound for the quantum dimensions of simple objects, provided the level  $k$  is not smaller than three,

$$\mathcal{D}_\lambda^{(k)} \geq \mathcal{D}_{\Lambda^1}^{(3)} = \frac{[3]_3[6]_3}{[1]_3[4]_3} > \sqrt{3}$$

and deduce that there are no Kramers-Wannier dualities for WZW theories based on  $E_6$ .

### 3.3 The series $C_r$

The Picard group of WZW theories based on the untwisted affine Lie algebra  $C_r^{(1)}$  is cyclic of order two; a duality class must be a fixed point of quantum dimension  $\sqrt{2}$ . It follows from the values of the Coxeter labels that the level  $k$  must be even for odd rank  $r$ ; for even rank there is no restriction on the level.

The isomorphism of complex simple Lie algebras  $C_1 \cong A_1$  allows us to assume that  $r \geq 2$ . For  $r \geq 2$  and  $k = 1$  or  $k + r \geq 6$  the second minimal quantum dimension is given [11] by

$$\mathcal{D}_\lambda^{(k,r)} \geq \mathcal{D}_{\Lambda^1}^{(k,r)} = \frac{\lfloor \frac{r}{2} \rfloor_k \lfloor r+1 \rfloor_k}{\lfloor \frac{1}{2} \rfloor_k \lfloor \frac{r+1}{2} \rfloor_k}.$$

Lemma 2.3.1 states  $\mathcal{D}_{\Lambda^1}^{(k,r)} \geq \mathcal{D}_{\Lambda^1}^{(1,r)}$  with equality of  $k = 1$  and we compute

$$\mathcal{D}_{\Lambda^1}^{(1,r)} = \frac{\lfloor 1 \rfloor_1 \lfloor \frac{r}{2} \rfloor_1}{\lfloor \frac{1}{2} \rfloor_1 \lfloor \frac{r+1}{2} \rfloor_1} \geq \sqrt{2},$$

with equality for  $r = 2$ . Indeed, the tensor category for  $(C_2)_1 \cong (B_2)_1$  has Ising fusion rules and therefore displays a Kramers-Wannier duality.

For  $k + r < 6$  second minimal quantum dimensions are

$$\begin{aligned} \mathcal{D}_{2\Lambda^1}^{(2,2)} &= \frac{\lfloor \frac{5}{2} \rfloor_2 \lfloor 1 \rfloor_2}{\lfloor 1 \rfloor_2 \lfloor 2 \rfloor_2} = 2 > \sqrt{2} \\ \mathcal{D}_{\Lambda^1}^{(3,2)} &= \frac{\lfloor 1 \rfloor_3 \lfloor 3 \rfloor_3}{\lfloor \frac{1}{2} \rfloor_3 \lfloor \frac{3}{2} \rfloor_3} = 1 + \sqrt{3} > \sqrt{2} \\ \mathcal{D}_{\Lambda^1}^{(2,3)} &= \frac{\lfloor \frac{3}{2} \rfloor_2 \lfloor 4 \rfloor_2}{\lfloor \frac{1}{2} \rfloor_2 \lfloor 2 \rfloor_2} = 1 + \sqrt{3} > \sqrt{2}. \end{aligned}$$

As a consequence, the only duality class occurs for  $C_2$  at level 1.

### 3.4 The series $B_r$

Again the Picard group is cyclic of order two so that we are led to classify fixed points of quantum dimension  $\sqrt{2}$ .

Because of the isomorphisms  $C_1 \cong A_1$  and  $C_2 \cong B_2$  of finite-dimensional complex Lie algebras, we restrict ourselves to  $r \geq 3$ . For  $r \geq 3$  and  $k \geq 4$  or  $k = 2$  the second minimal quantum dimension is given [11] by

$$\mathcal{D}_{\Lambda^1}^{(k,r)} = \frac{\lfloor r + \frac{1}{2} \rfloor_k \lfloor 2r - 1 \rfloor_k}{\lfloor 1 \rfloor_k \lfloor r - \frac{1}{2} \rfloor_k}$$

and with the monotonicity lemma 2.3.1 we obtain

$$\mathcal{D}_{\Lambda^1}^{(k,r)} > \mathcal{D}_{\Lambda^1}^{(2,r)} = \frac{\lfloor 2 \rfloor_2 \lfloor r + \frac{1}{2} \rfloor_2}{\lfloor 1 \rfloor_2 \lfloor r - \frac{1}{2} \rfloor_2} = 2.$$

This excludes duality classes for all levels  $k \geq 4$  and  $k = 2$ .

At level  $k = 1$ , we find Ising fusion rules and thus the unique duality class  $\Lambda^r$ . At level  $k = 3$ , the second lowest quantum dimension is assumed for the weight  $3\Lambda^r$  [11]. For its quantum dimension, we find

$$\mathcal{D}_{3\Lambda^r}^{(3,r)} = \frac{\lfloor 2 \rfloor_3}{\lfloor r \rfloor_3} \prod_{l=1}^r \frac{\lfloor 2l - 1 \rfloor_3}{\lfloor l - \frac{1}{2} \rfloor_3} = \frac{\lfloor 2 \rfloor_3}{\lfloor r \rfloor_3} \prod_{l=1}^r \cos\left(\pi \frac{l - \frac{1}{2}}{2r + 2}\right),$$

which is strictly larger than  $\sqrt{2}$  for all ranks  $r \geq 2$ : We consider the case of even and odd rank separately; for even rank we find

$$\mathcal{D}_{3\Lambda^r}^{(3,r)} = \frac{\lfloor 1 \rfloor_3 \lfloor 2 \rfloor_3}{\lfloor \frac{1}{2} \rfloor_3 \lfloor \frac{3}{2} \rfloor_3} \prod_{l=2}^{\frac{r}{2}} \frac{\lfloor 2l - 1 \rfloor_3 \lfloor 2l - 1 \rfloor_3}{\lfloor 2l - \frac{1}{2} \rfloor_3 \lfloor 2l - \frac{3}{2} \rfloor_3} \frac{\lfloor r + 1 \rfloor_3}{\lfloor r \rfloor_3};$$

with the notation  $\lceil x \rceil_k := \cos\left(\frac{\pi x}{k+g^\vee}\right)$  this is equal to

$$\mathcal{D}_{3\Lambda^r}^{(3,r)} = \frac{\lfloor 1 \rfloor_3 \lfloor 2 \rfloor_3}{\lfloor \frac{1}{2} \rfloor_3 \lfloor \frac{3}{2} \rfloor_3} \prod_{l=2}^{\frac{r}{2}} \frac{1 - \lceil 4l - 2 \rceil_3}{\lceil 1 \rceil_3 - \lceil 4l - 2 \rceil_3} \frac{\lfloor r + 1 \rfloor_3}{\lfloor r \rfloor_3}.$$

Since the arguments of the sine- and cosine-functions are all smaller than  $\pi/2$ , all quotients are bigger than one. The first quotient is strictly larger than  $\sqrt{3}$ , because  $\frac{\frac{\pi}{2}}{2r+2} < \frac{\pi}{6}$  for any value of  $r$ . Therefore, the quantum dimension  $\mathcal{D}_{3\Lambda^r}^{(3,r)}$  is strictly larger than  $\sqrt{2}$ .

We proceed in an analogous way for odd rank to find

$$\mathcal{D}_{3\Lambda^r}^{(3,r)} = \frac{\lfloor 1 \rfloor_3 \lfloor 2 \rfloor_3}{\lfloor \frac{1}{2} \rfloor_3 \lfloor \frac{3}{2} \rfloor_3} \prod_{l=2}^{\frac{r-1}{2}} \frac{1 - \lceil 4l - 2 \rceil_3}{\lceil 1 \rceil_3 - \lceil 4l - 2 \rceil_3} \frac{\lfloor r \rfloor_3}{\lfloor r - \frac{1}{2} \rfloor_3} \geq \sqrt{3}.$$

We conclude that there are no duality classes for levels  $k \geq 2$ .

### 3.5 The series $D_r$

The structure of the Picard group depends on whether the rank is odd or even. For even rank, it is isomorphic to the Kleinian four group  $\mathbb{Z}_2 \times \mathbb{Z}_2$  with generators  $k\Lambda^{r-1}$  and  $k\Lambda^r$ . Accordingly, we have to consider this group as

well as its three cyclic subgroups of order two as possible stabilizers. For odd rank, the Picard group is cyclic of order four with  $k\Lambda^{r-1}$  or  $k\Lambda^r$  as possible generators. The possible stabilizers are then the full Picard group and the cyclic group of order two generated by  $k\Lambda^1$ .

For  $r \geq 4$  and  $k \geq 3$  the second minimal quantum dimension is given by

$$\mathcal{D}_{\Lambda^1}^{(k,r)} = \frac{[r]_k [2r-2]_k}{[1]_k [r-1]_k}$$

and a lower bound is by Lemma 2.3.1

$$\mathcal{D}_{\Lambda^1}^{(k,r)} > \mathcal{D}_{\Lambda^1}^{(2,r)} = \frac{[2]_2 [r]_2}{[1]_2 [r-1]_2} = 2.$$

We deduce that for  $k \geq 3$  no duality classes for any stabilizer exist: for cyclic stabilizers of order two this is immediate. Duality objects whose stabilizer is the full Picard group must be fixed points under all four simple currents. They can only occur at even levels; but for level equal to four and higher, by the monotonicity properties of the quantum dimensions, no simple object of quantum dimension two exists.

The remaining case is level  $k = 2$ . It is convenient to treat  $D_4$  separately: in this case, simple objects have quantum dimension equal to one or two. There is a single duality class with highest weight  $\Lambda^2$  which is a fixed point under the whole Picard group. We find a single duality class.

For rank greater or equal to five and level  $k = 2$ , there are four simple currents, four simple objects of quantum dimension strictly bigger than two and all simple objects with weights  $\Lambda^1$ ,  $\Lambda^{r-1} + \Lambda^r$  and  $\Lambda^i$  with  $i = 2, \dots, r-2$  have quantum dimension 2. Among these weights, however, only  $\Lambda^{r/2}$  for even rank is a fixed point under the whole Picard group and provides a duality class with the Picard group  $\mathbb{Z}_2 \times \mathbb{Z}_2$  as the stabilizer.

### 3.6 The series $A_r$

In this case, we can use Theorem 2.2.1 with  $U_\mu$  equal to the defining  $(r+1)$ -dimensional irreducible representation with highest weight  $\Lambda^1$ . Monodromy classes of WZW theories are in correspondence with conjugacy classes of representations. For  $A_r$  the defining representation is a generator of the group of conjugacy classes of representations and hence has nontrivial monodromy character. For the tensor product with the dual simple object we find

$$\Lambda^1 \otimes (\Lambda^1)^\vee = \Lambda^1 \otimes \Lambda^{r-1} \cong 0 \oplus (\Lambda^1 + \Lambda^r).$$

The second simple object in the direct sum is an invertible object only for  $r = 1$  and  $k = 2$ . Thus, the only duality class appears for  $(A_1)_2$  which is known to have Ising fusion rules.

## 4 Dualities for (super-)minimal models

### 4.1 Virasoro minimal models

Nonsupersymmetric Virasoro minimal models can be obtained by the coset construction [12] from the diagonal embedding

$$(A_1)_{k+1} \hookrightarrow (A_1)_k \oplus (A_1)_1 .$$

To describe isomorphism classes of simple objects, consider triples  $\Phi_t^{l\ s}$ , where  $l \in \{0, \dots, k\}$ ,  $s \in \{0, 1\}$  and  $t \in \{0, \dots, k+1\}$ , subject to the condition

$$l + s - t = 0 \bmod 2 . \quad (14)$$

On such triples, simple objects are equivalence classes of the relation

$$\Phi_t^{l\ s} \sim \Phi_{k+1-t}^{k-l\ 1-s} .$$

The decomposition of the tensor product of simple objects into a direct sum of simple objects is given by

$$\Phi_{t_1}^{l_1\ s_1} \otimes \Phi_{t_2}^{l_2\ s_2} \cong \bigoplus_{l_3=|l_1-l_2|}^{l_{\max}} \bigoplus_{t_3=|t_1-t_2|}^{t_{\max}} \Phi_{t_3}^{l_3\ s_3}$$

with  $l_{\max} = \min(l_1 + l_2, 2k - l_1 - l_2)$  and  $t_{\max} = \min(t_1 + t_2, 2k + 2 - t_1 - t_2)$ , and where the indices are required to fulfill

$$l_1 + l_2 + l_3 = 0 \bmod 2 \quad \text{and} \quad t_1 + t_2 + t_3 = 0 \bmod 2 .$$

The selection rule (14) fixes the value of  $s_3$  in terms of  $l_3$  and  $t_3$ .

The simple currents apart from the isomorphism class of the tensor unit are

$$\begin{aligned} \Phi_{k+1}^{0\ 0} &\sim \Phi_0^{k\ 1} && \text{for } k \text{ odd} \\ \Phi_{k+1}^{0\ 1} &\sim \Phi_0^{k\ 0} && \text{for } k \text{ even,} \end{aligned}$$

hence  $\text{Pic}(\mathcal{C})$  is cyclic of order two. The monodromy characters are products of monodromy characters for WZW theories based on  $A_1$ .

The only simple objects that can be used to apply Theorem 2.2.1 are  $\Phi_0^{1\ 1}$ ,  $\Phi_1^{0\ 1}$  and  $\Phi_k^{0\ 1}$  for odd level  $k$  and  $\Phi_{k+1}^{1\ 0}$  for even level  $k$ . The tensor product with the dual object is

$$\begin{aligned} \Phi_0^{1\ 1} \otimes \Phi_0^{1\ 1} &\cong \Phi_0^{0\ 0} \oplus \Phi_0^{2\ 0} && \text{for level } k \geq 2 \\ \Phi_1^{0\ 1} \otimes \Phi_1^{0\ 1} &\cong \Phi_0^{0\ 0} \oplus \Phi_2^{0\ 0} \\ \Phi_k^{0\ 1} \otimes \Phi_k^{0\ 1} &\cong \Phi_0^{0\ 0} \oplus \Phi_2^{0\ 0} && \text{for odd level} \\ \Phi_{k+1}^{1\ 0} \otimes \Phi_{k+1}^{1\ 0} &\cong \Phi_0^{0\ 0} \oplus \Phi_0^{2\ 0} && \text{for even level.} \end{aligned}$$

$\Phi_2^{0\ 0}$  is a simple current only for  $k = 1$ ; this case is indeed the Ising model with its well-known Kramers-Wannier duality. The primary field  $\Phi_0^{2\ 0}$  is a simple current only for  $k = 2$ . In this case, we recover the known Kramers-Wannier duality of the tricritical Ising model.

## 4.2 $N = 1$ super-Virasoro minimal models

The  $N = 1$  superconformal minimal models have the following coset description [12] based on the diagonal embedding

$$(A_1)_{k+2} \hookrightarrow (A_1)_k \oplus (A_1)_2 .$$

To describe isomorphism classes of simple objects, consider triples  $\Phi_t^{l\ s}$  with where  $l \in \{0, \dots, k\}$ ,  $s \in \{0, 1, 2\}$  and  $t \in \{0, \dots, k+2\}$ , subject to the requirement

$$l + s - t = 0 \bmod 2 .$$

For odd level  $k$ , representatives for simple objects are labelled by equivalence classes of triples under the equivalence relation

$$\Phi_t^{l\ s} \sim \Phi_{k+2-t}^{k-l\ 2-s} .$$

For even level, the same holds with the exception that there are two simple objects corresponding to the triple  $\Phi_{\frac{k}{2}+1}^{\frac{k}{2}\ 1}$ . This phenomenon is called “fixed point resolution” in the physics literature (see e.g. [17] or [18] for a review).

The tensor products with no fixed points involved are given by

$$\Phi_{t_1}^{l_1\ s_1} \otimes \Phi_{t_2}^{l_2\ s_2} \cong \bigoplus_{l_3=|l_1-l_2|}^{l_{\max}} \bigoplus_{s_3=|s_1-s_2|}^{s_{\max}} \bigoplus_{t_3=|t_1-t_2|}^{t_{\max}} \Phi_{t_3}^{l_3\ s_3} ,$$

where the indices are required to obey

$$\begin{aligned} l_1 + l_2 + l_3 &= 0 \bmod 2 \\ s_1 + s_2 + s_3 &= 0 \bmod 2 \\ t_1 + t_2 + t_3 &= 0 \bmod 2 \end{aligned}$$

with  $l_{\max} = \min(l_1 + l_2, 2k - l_1 - l_2)$ ,  $s_{\max} = \min(s_1 + s_2, 4 - s_1 - s_2)$  and  $t_{\max} = \min(t_1 + t_2, 2k + 4 - t_1 - t_2)$ .

The simple currents are

$$\begin{aligned} \Phi_0^{0\ 0} &\sim \Phi_{k+2}^{k\ 2} \\ \Phi_0^{0\ 2} &\sim \Phi_{k+2}^{k\ 0} \\ \Phi_{k+2}^{0\ 0} &\sim \Phi_0^{k\ 2} \quad \text{for } k \text{ even} \\ \Phi_{k+2}^{0\ 2} &\sim \Phi_0^{k\ 0} \quad \text{for } k \text{ even,} \end{aligned}$$



hence  $\text{Pic}(\mathcal{C}) \cong \mathbb{Z}_2$  for odd level  $k$  and  $\text{Pic}(\mathcal{C}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  for even level  $k$ .

We start with the discussion of dualities with cyclic stabilizer of order two. The duality classes have to be fixed points of quantum dimension  $\sqrt{2}$  under the action of a nontrivial simple current of order two. The quantum dimension of  $\Phi_s^{l\ m}$  is a product of the  $A_1$  quantum dimensions.

We consider first fixed points of the simple current  $\Phi_0^{0\ 2}$ : in this case  $s = 1$  so that this  $A_1$  summand already contributes a multiplicative factor  $\sqrt{2}$  to the quantum dimension; the other labels  $l, t$  have to correspond to  $A_1$ -simple currents:  $l \in \{0, k\}$  and  $t \in \{0, k+2\}$ . This is excluded by the selection rule  $l + s + t = 0 \pmod{2}$  for even level  $k$ . For odd level  $k$  we find a single duality class  $\Phi_{k+2}^{0\ 1}$  with a cyclic stabilizer of order two generated by  $\Phi_0^{0\ 2}$ .

For even  $k$  we have to consider fixed points under the action of the simple current  $\Phi_{k+2}^{0\ 0}$  as well. In this case, the relevant  $A_1$  constituent has level greater or equal to three so that already this part makes a multiplicative contribution to the quantum dimensions strictly bigger than  $\sqrt{2}$ . Thus, the cyclic group of order two generated by  $\Phi_{k+2}^{0\ 0}$  for  $k$  even never occurs as a stabilizer.

We finally consider fixed points of the simple current  $\Phi_0^{k\ 0}$  for even level  $k$ . By the same arguments, quantum dimension  $\sqrt{2}$  for a fixed point can only be achieved for  $k = 2$  and for  $\Phi_t^{1\ s}$  with  $s, t$  describing simple currents. Such fixed points are, however, excluded by the parity rule  $l + m + s = 0 \pmod{2}$ .

For even level  $k$ , the full Picard group could appear as a stabilizer as well. We should therefore find all fixed points of quantum dimension two under the action of all four simple currents. Only the two simple objects arising in the “fixed point resolution” of  $\Phi_{(k+2)/2}^{k/2\ 1}$  qualify. Indeed, a computation with KAC [19] shows that quantum dimension 2 is achieved for level  $k = 2$ . Since the monotonicity lemma for WZW theories implies, by the coset construction, the same monotonicity properties in  $k$ , this is the only relevant case. A computation of the fusion rules, e.g. again with KAC, however shows that the two simple objects arising in the fixed resolution are not fixed points of all four simple currents. Hence, for even  $k$ , there is no duality object whose stabilizer is the full Picard group.

### 4.3 $N = 2$ super-Virasoro minimal models

A coset realization for  $N = 2$  superconformal minimal models is based on the embedding

$$u(1)_{2(k+2)} \hookrightarrow (A_1)_k \oplus u(1)_4,$$

where  $(u_1)_N$  stands for the modular tensor category with  $K_0((u_1)_N) = \mathbb{Z}/N\mathbb{Z}$  (for details, see e.g. Section 2.5.1 of [9]). To describe simple ob-

jects, consider triples  $\Phi_t^{l\ s}$  with  $l \in \{0, \dots, k\}$ ,  $s \in \mathbb{Z}/4\mathbb{Z}$  and  $t \in \mathbb{Z}/2(k+2)\mathbb{Z}$ , subject to the condition

$$l + s - t = 0 \bmod 2 .$$

Isomorphism classes of simple objects can be labelled by equivalence classes of the equivalence relation

$$\Phi_t^{l\ s} \sim \Phi_{t \pm (k+2)}^{k-l\ s \pm 2} .$$

The tensor products read

$$\Phi_{t_1}^{l_1\ s_1} \otimes \Phi_{t_2}^{l_2\ s_2} \cong \bigoplus_{l_3=|l_1-l_2|}^{l_{\max}} \Phi_{(t_1+t_2)}^{l_3\ (s_1+s_2)}$$

with  $l_{\max} = \min(l_1+l_2, 2k-l_1-l_2)$ , where the index  $l$  must obey the selection rule

$$l_1 + l_2 + l_3 = 0 \bmod 2 .$$

We can apply Theorem 2.2.1 to the isomorphism class of simple objects  $\Phi_0^{1\ 1}$  for which the relevant tensor product reads for  $k \geq 2$ :

$$\Phi_0^{1\ 1} \otimes \Phi_0^{1\ -1} \cong \Phi_0^{0\ 0} \oplus \Phi_0^{2\ 0} .$$

Unless  $k = 2$ , the second simple object is not a simple current. For  $k = 2$  we find 16 simple currents and 8 duality classes  $\Phi^1_\cdot$  which all have a cyclic stabilizer generated by  $\Phi_0^{2\ 0}$ , the worldsheet supercurrent.

The  $N = 2$  superconformal theory with  $k = 1$  has Virasoro central charge  $c = 1$  and is equivalent to a free boson compactified on a circle of appropriate radius. As a consequence, all isomorphism classes of simple objects are simple currents in this case and there are no duality objects.

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